Université de Toulouse
Département de Mathématiques M2 RI

Basic course A6
An introduction to the theoretical and numerical analysis of scalar conservation laws

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The goal of this course is to introduce the audience to the study of partial differential equations of hyperbolic type and more precisely of scalar conservation laws of the form

$$\partial_t \rho(t,x) + \operatorname{div} F(t,x,\rho(t,x)) = 0,$$

for $t > 0$ and $x \in \Omega$ where Ω is an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^\star$) and where $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ (with $n \in \mathbb{N}^\star$) is a given vector field function.
We will particularly concentrate on the linear case corresponding to $F(t,x,\rho) = \rho\, v(t,x)$ with $v : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and on the one-dimensional (1-D) nonlinear case where $F(t,x,\rho) = f(\rho)$ with $f : \mathbb{R} \to \mathbb{R}$.

Prerequisites: Differential calculus and differential equations. Basic functional analysis. Lebesgue integration theory.

1. Modelling of transport phenomena
   - Some examples in the 1-D case (traffic flow, gas dynamics, ...)
   - The multi-D case : Reynolds transport theorem. Continuity and transport equations
2. The linear transport equation
   - Smooth solutions by the method of characteristics
   - Weak solutions of the linear transport equation. Jump conditions. Existence and uniqueness
   - Finite Difference and Finite volume schemes in 1-D : construction, consistency, stability, convergence.
   - Extensions to the multi-D setting.
3. Nonlinear conservation laws in 1-D
   - Using the characteristics : local in time well-posedness of smooth solutions, finite time singularities, nonlinear regularization process
   - Study of weak solutions : Rankine-Hugoniot conditions, global in time existence, non uniqueness
   - Entropy solutions : physical motivations, definition and characterizations (Lax conditions, Oleinik conditions),
   - Krushkov entropies. Uniqueness of the entropy solution by the doubling variables technique. Krushkov theorem.

- Riemann problems : definition, shocks and rarefaction waves
- Finite volume methods : monotone fluxes, TVD schemes. Convergence in 1-D towards an entropy solution for BV data

4. Possible extensions

   - Few words concerning hyperbolic systems
   - Finite volume schemes for scalar conservation laws in any dimension
   - Boundary conditions

References

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[3] Lawrence C. EVANS, Partial Differential Equations. 2nd Edition, Department of Mathematics, University of California, Berkeley, American Mathematical Society, 2010.

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[6] Eleuterio F. TORO, Riemann solvers and numerical methods for fluid dynamics. A practical introduction, Springer-Verlag, Berlin, third edition, 2009.